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NOTE

On a Discrete Kolmogorov Criterion

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In this note we consider best approximation in 1-norm from subspaces of real n-dimensional spaces. Some discrete Kolmogorov-type criteria characterizing elements of best approximation are verified. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Consider $l_1(n)$, i.e., $\mathbb{R}^n$ endowed with the 1-norm $\|\cdot\|_1$, and let U be an r-dimensional linear subspace of $\mathbb{R}^n$, $0<r<n$. Let us denote the metric projection

$$P_U\colon \mathbb{R}^n \to 2^U,$$
$$x \mapsto \{u \in U : \|x-u\|_1 = \operatorname{dist}(x; U)\},$$

where $\operatorname{dist}(x; U) = \min_{u\in U} \|x-u\|_1$ is the distance of x to the subspace U.

The famous *Kolmogorov criterion* characterizes all best approximants:

Classical Kolmogorov criterion. *For $x \in \mathbb{R}^n$, $u^* \in U$, and $Z_{x-u^*} := \{1, 2, ..., n\} \backslash \operatorname{supp}(x-u^*)$ we have*

$$u^* \in P_U(x) \Leftrightarrow \sum_{i \notin Z_{x-u^*}} sgn(x_i - u^*) \cdot u_i \leqslant \sum_{i \in Z_{x-u^*}} |u_i| \qquad \text{for all} \quad u \in U. \tag{1}$$

Here supp y denotes the support of the vector $y \in \mathbb{R}^n$, the set of not vanishing components of y.

It goes back to Gauß that for each x a best approximant in U exists which interpolates x in at least r components, i.e., for each $x \in \mathbb{R}^n$ there exists a $u^* \in P_U(x)$ and pairwise distinct indices $v_1, ..., v_r \in \{1, 2, ..., n\}$ with

$$x_{v_p} = u^*_{v_p}, \qquad 1 \leqslant \rho \leqslant r.$$

Most algorithms which determine a best approximant of x in U are based on this fact. For this particular purpose the Kolmogorov criterion has been specialized and discretized such that (1) needs to be checked for only finitely

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many points u in U, see Bloomfield and Steiger [1] or Pinkus [5] for details.
Pinkus [5, Prop. 7.6] also gives a proof of this fact by contradiction.

First we give a discretized formulation of the restricted Kolmogorov criterion. The criterion, however, characterizes only those best approximants of a given vector x which interpolates x in at least r components:

DISCRETE RESTRICTED KOLMOGOROV CRITERION. *For* $x \in \mathbb{R}^n$ *and* $u^* \in U$ *with* $Z = Z_{x-u^*} := \{1, 2, ..., n\} \backslash \operatorname{supp}(x - u^*)$, card $Z \geqslant r$ *and* rank$[\begin{smallmatrix} U \\ Z \end{smallmatrix}] = r$, *we have*

$$u^* \in P_U(x) \Leftrightarrow \sum_{i \notin Z_{x-u^*}} sgn(x_i - u_i^*) \cdot u_i \leqslant \sum_{i \in Z_{x-u^*}} |u_i|$$

for all elementary vectors $u \in U$ *for which an index set* $J \subset Z_{x-u^*}$ *exists with*

$$\operatorname{card} J = r, \quad \operatorname{rank} \begin{bmatrix} U \\ J \end{bmatrix} = r, \qquad and \qquad \operatorname{card}(\operatorname{supp} u \cap J) = 1.$$

Let $U^{\perp}$ denote the orthogonal complement of U, and

$$\begin{bmatrix} U \\ J \end{bmatrix} = \begin{bmatrix} U \\ v_1 \cdots v_s \end{bmatrix} = \begin{bmatrix} v_{v_1}^{(1)} & \cdots & v_{v_1}^{(r)} \\ \vdots & & \vdots \\ v_{v_s}^{(1)} & \cdots & v_{v_s}^{(r)} \end{bmatrix},$$

where $v^{(1)}, ..., v^{(r)}$ is an arbitrary basis of U and $J = \{v_1, ..., v_s\}$.

Elementary vectors of a subspace are vectors with minimal support in the following sense: The vector $u \in U$ is elementary if and only if $w \in U$ with supp $w \subset$ supp u implies $w = \lambda \cdot u$ for some $\lambda \in \mathbb{R}$, see Rockafellar [6, 203ff].

In general, however, there exist best approximants which interpolate the given vector x in less than r components. For these approximants the discrete restricted Kolmogorov criterion above is not applicable. Nevertheless, in this general situation the Kolmogorov criterion also allows a discrete formulation which unifies both advantages of the classical and the discrete restricted criterion. In characterizes best approximants, in general, and requires only to check (1) for elementary vectors of U:

THEOREM (Discrete Kolmogorov criterion). *For* $x \in \mathbb{R}^n$, $u^* \in U$, *and* $Z_{x-u^*} := \{1, 2, ..., n\} \backslash \operatorname{supp}(x - u^*)$ *we have*

$$u^* \in P_U(x) \Leftrightarrow \sum_{i \notin Z_{x-u^*}} \operatorname{sgn}(x_i - u_i^*) \cdot u_i \leqslant \sum_{i \in Z_{x-u^*}} |u_i|$$

for all elementary vectors u *in* U.

Note, if an elementary vector $u \in U$ satisfies the inequality above, so does the elementary vector λu, $\lambda > 0$. In other words, it is sufficient to check the inequality above for all (finitely many) extremal points of the polyhedron Q, see Lemma 1(ii), below.

2. PROOF FOR THE DISCRETE KOLMOGOROV CRITERIA

In this section we first formulate a lemma which characterizes elementary vectors from different points of view. Then we prove the theorem. Its proof needs a technical lemma on elementary vectors. Finally, we give a direct proof of Theorem 1.

The following lemma states equivalent characterizations of elementary vectors:

LEMMA 1. *For a vector u in U the following statements are equivalent*:

(i) $u \in U$ *is an elementary vector*;

(ii) $u/\|u\|_1 \in \operatorname{ext} Q$, *where* $Q = U \cap \overline{b_1^1(0)}$ *denotes the intersection of* U *with the closed unit ball in* $l_1(n)$; Q *is an* r*-dimensional closed convex and symmetric polyhedron*;

(iii) $$\operatorname{rank}\begin{bmatrix} U^\perp \\ \operatorname{supp} u \end{bmatrix} = \operatorname{rank}\begin{bmatrix} U^\perp \\ \mu_1 \cdots \mu_{m-1} \end{bmatrix} = \operatorname{card} \operatorname{supp} u - 1$$

for all $\{\mu_1, ..., \mu_{m-1}\} \subset \operatorname{supp} u$ *and* $\operatorname{card} \operatorname{supp} u = m$;

(iv) $$\operatorname{rank}\begin{bmatrix} U \\ \operatorname{supp}^c u \end{bmatrix} = r - 1 \qquad \text{and} \qquad \operatorname{rank}\begin{bmatrix} U \\ \operatorname{supp}^c u \cup \{\mu\} \end{bmatrix} = r$$

for all $\mu \in \operatorname{supp} u$.

Here $\operatorname{supp}^c u := \{1, 2, ..., n\} \backslash \operatorname{supp} u$ denotes the complementary index set of the support set of u.

For the equivalence of (i), (ii), and (iii) see [2], while the equivalence of (iii) and (iv) follows from the following relation, see [3]:

$$\operatorname{rank}\begin{bmatrix} U \\ \{1, 2, ..., n\} \backslash I \end{bmatrix} = \operatorname{rank}\begin{bmatrix} U^\perp \\ I \end{bmatrix} + r - \operatorname{card} I \qquad \text{for any} \quad I \subset \{1, 2, ..., n\}.$$

Rockafellar [6] defined elementary vectors in terms of their supports. The support of an elementary vector uniquely determines it up to a scalar factor. The geometric characterization (ii) of Lemma 1 eliminates this degree of freedom by normalizing the elementary vector $u \in U: \|u\|_1 = 1$. All normalized elementary vectors lie on the $l_1(n)$-unit sphere and form exactly the set of vertices of the polyhedron $Q = U \cap \overline{b_1^1(0)}$. Clearly, there are only finitely many extremal points of Q and in this sense there are only finitely many elementary vectors in the subspace, which are also called elementary directions. The collection of all elementary directions of a linear subspace spans the subspace; in general, however, the elementary directions are linearly dependent. Moreover, each elementary vector u of the subspace U has at least $r-1$ zero components, and, consequently, at most $n-r+1$ components which do not vanish.

From Lemma 1(iv) it follows that each elementary vector u in U has a representation $u=\sum_{\rho=1}^{r} \alpha_\rho v^{(\rho)}$, where $\alpha_1, ..., \alpha_r$ is a non-vanishing solution of the homogeneous system of linear equalities $\sum_{\rho=1}^{r} \alpha_\rho v_\nu^{(\rho)}=0$, $\nu \in \operatorname{supp}^c u$.

On the other hand, it follows from Lemma 1(iii) that each elementary vector u in U is a normal vector of the hyperplane in $\mathbb{R}^{\operatorname{card} \operatorname{supp} u}$ spanned by the columns of the matrix $[{}^{U^\perp}_{\operatorname{supp} u}]$ which has rank deficiency one. Assume $\operatorname{card} \operatorname{supp} u=m$ and $\operatorname{rank}[{}^{w^{(1)} \cdots w^{(m-1)}}_{\operatorname{supp} u}]=m-1$ where $w^{(1)}, ..., w^{(m-1)} \in U^\perp$, then the elementary vector u is given by the generalized vector (or cross) product of the column vectors of the $m \times (m-1)$-matrix above. For more information on elementary vectors see [2] and [4].

Lemma 2. *For each vector u in U there exist elementary vectors* $u^{(1)}, ..., u^{(l)} \in U$ *and scalars* $\alpha_1, ..., \alpha_l \geqslant 0$ *with*

$$u=\sum_{\lambda=1}^{l} \alpha_\lambda u^{(\lambda)} \qquad \text{and} \qquad u_i^{(\lambda)} \cdot u_i \geqslant 0, \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant \lambda \leqslant l.$$

Proof. Assume without loss of generality $\|u\|_1=1$. Then u belongs to a face of the relative boundary of the polyhedron Q. Let $u^{(1)}, ..., u^{(l)}$ be the extremal points of this face. Consequently $u^{(1)}, ..., u^{(l)}$ are also extremal points of Q, and hence elementary vectors in U.

Remember that each hyper-face of the unit ball $\overline{b_1^1(0)}$ belongs to a closed orthant of $\mathbb{R}^n$, and so does each face of Q. We have $u_i^{(\lambda)} \cdot u_i \geqslant 0$ for all $1 \leqslant i \leqslant n$ and all $1 \leqslant \lambda \leqslant l$. Finally, u is a convex combination of the extremal points of the face of Q where u belongs to; i.e., there exist $\alpha_1, ..., \alpha_l \geqslant 0$ with $u=\sum_{\lambda=1}^{l} \alpha_\lambda u^{(\lambda)}$. ∎

Proof of the discrete Kolmogorov criterion. "$\Rightarrow$": It follows from the classical Kolmogorov criterion.

"$\Leftarrow$": Let u^* be in $\mathbb{R}^n$, and $u \in U$. By Lemma 2 there exist elementary vectors $u^{(1)}, ..., u^{(l)}$ in U and scalars $\alpha_1, ..., \alpha_l \geqslant 0$ with $u=\sum_{\lambda=1}^{l} \alpha_\lambda u^{(\lambda)}$ and $u_i^{(\lambda)} \cdot u_i \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant \lambda \leqslant l$. By assumption we have for $1 \leqslant \lambda \leqslant l$

$$\sum_{i \notin Z_{x-u^*}} sgn(x_i - u_i^*) \cdot u_i^{(\lambda)} \leqslant \sum_{i \in Z_{x-u^*}} |u_i^{(\lambda)}|,$$

and hence

$$\sum_{i \notin Z_{x-u^*}} sgn(x_i - u_i^*) \cdot \sum_{\lambda=1}^{l} \alpha_\lambda u_i^{(\lambda)} \leqslant \sum_{i \in Z_{x-u^*}} \sum_{\lambda=1}^{l} \alpha_\lambda |u_i^{(\lambda)}|$$

$$= \sum_{i \in Z_{x-u^*}} \left| \sum_{\lambda=1}^{l} \alpha_\lambda u_i^{(\lambda)} \right|.$$

The last equality holds because of all summands have the same sign or vanish. So the classical Kolmogorov criterion is satisfied and hence u^* is in $P_U(x)$. ▮

A direct proof of the discrete restricted Kolmogorov criterion is based on the argument given above and the following lemma. For this we need some notation. Let be $x \in \mathbb{R}^n$, $u^* \in P_U(x)$, and $Z := \{1, 2, ..., n\} \setminus \mathrm{supp}(x - u^*)$ with card $Z = s \geqslant r$ and $\mathrm{rank}[{}^U_Z] = r$. The orthogonal projection Π_Z: $\mathbb{R}^n \to \mathbb{R}^s$, $y \mapsto \bar{y}$ with $\bar{y}_i = y_i$ for $i \in Z$ maps the r-dimensional linear subspace U to the r-dimensional linear subspace $\bar{U}$ of $\mathbb{R}^s$, and the basis vectors $v^{(1)}$, ..., $v^{(r)}$ of U to the basis vectors $\bar{v}^{(\rho)} = \Pi_Z(v^{(\rho)})$ of $\bar{U}$, $1 \leqslant \rho \leqslant r$.

LEMMA 3. *Under the notation given above, $\bar{u}$ is an elementary vector in $\bar{U}$ if and only if $\bar{u} = \Pi_Z(u)$ for an elementary vector u in U with*

$$1 \leqslant \mathrm{card}(\mathrm{supp}\, u \cap Z) \qquad \textit{and} \qquad \mathrm{rank}\begin{bmatrix} U \\ \mathrm{supp}^c\, u \cap Z \end{bmatrix} \geqslant r - 1.$$

Proof. "⇒": For $\bar{u} \in \bar{U}$ we have a unique representation $\bar{u} = \sum_{\rho=1}^{r} \beta_\rho \bar{v}^{(\rho)}$, and hence $u = \sum_{\rho=1}^{r} \beta_\rho v^{(\rho)}$ and $\Pi_z(u) = \bar{u}$. Let w be in U with $\mathrm{supp}\, w \subset \mathrm{supp}\, u$. We have $\mathrm{supp}\, \Pi_z(w) \subset \mathrm{supp}\, \Pi_z(u)$, and hence $\Pi_Z(w) = \lambda \cdot \Pi_Z(u)$ for a $\lambda \in \mathbb{R}$. By uniqueness of the representation in terms of a basis of $\bar{U}$ and U, respectively, it follows that $w = \lambda \cdot u$; thus u is elementary in U.

Since $\bar{u} \neq 0$ we have $1 \leqslant \mathrm{card}\, \mathrm{supp}\, \bar{u} = \mathrm{card}(\mathrm{supp}\, u \cap Z)$.

Moreover, since $\bar{u}$ is elementary, the system of linear equalities $\sum_{\rho=1}^{r} \gamma_\rho v_i^{(\rho)} = 0$, $i \in \{1, ..., s\} \setminus \mathrm{supp}\, \bar{u} = \mathrm{supp}^c\, u \cap Z$ has a one-dimensional set of solutions, and consequently, $\mathrm{rank}[{}^{U}_{\mathrm{supp}^c u \cap Z}] = r - 1$.

"⇐": The rank condition implies that there exists up to scalar multiplication only one vector $\bar{v}$ in $\bar{U}$ with $\mathrm{supp}\, \bar{v} \subset \mathrm{supp}^c\, u \cap Z = \mathrm{supp}\, \bar{z}$. From $0 \neq \bar{u} = \Pi_Z(u)$ we conclude $\bar{v} = \lambda \cdot \bar{u}$ for some $\lambda \in \mathbb{R}$, and hence $\bar{u}$ is elementary in $\bar{U}$. ▮

Proof the discrete restricted Kolmogorov criterion. We only need to prove "⇐": Let x be in $\mathbb{R}^n$ and u^* in U.

Consider u in U and $\tilde{u} := \Pi_Z(u)$. By Lemma 2 there exist elementary vectors $\bar{u}^{(1)}, ..., \bar{u}^{(l)}$ in $\bar{U} = \Pi_Z(U)$ and scalars $\alpha_1, ..., \alpha_l \geqslant 0$ with

$$\tilde{u} = \sum_{\lambda=1}^{l} \alpha_\lambda \bar{u}^{(\lambda)} \quad \text{and} \quad \bar{u}_i^{(\lambda)} \cdot \tilde{u}_i \geqslant 0, \qquad i \in Z, \quad 1 \leqslant \lambda \leqslant l.$$

Let us consider $\bar{u}^{(\lambda)}$ more closely, $1 \leqslant \lambda \leqslant l$: Since $\bar{u}^{(\lambda)}$ is elementary in $\bar{U}$, by Lemma 1(iv) there exist indices $v_1, ..., v_{r-1} \in \operatorname{supp}^c \bar{u}^{(\lambda)}$ such that

$$\operatorname{rank} \begin{bmatrix} U \\ \operatorname{supp}^c \bar{u}^{(\lambda)} \cup \{v_r\} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} U \\ J \end{bmatrix} = r$$

with $J = \{v_1, ..., v_r\}$. In particular, we have card $J = r$ and card(supp $\bar{u}^{(\lambda)} \cap J$) $= 1$. By Lemma 3 there exists an elementary vector $u^{(\lambda)}$ in U with $\bar{u}^{(\lambda)} = \Pi_Z(u^{(\lambda)})$. Clearly, for $u^{(\lambda)}$ the index set $J \subset Z$ satisfies card $J = r$, rank$[{}^U_J] = r$ and card(supp $u^{(\lambda)} \cap J) = 1$. Consequently,

$$\sum_{i \notin Z} sgn(x_i - u_i^*) \cdot u_i^{(\lambda)} \leqslant \sum_{i \in Z} |u_i^{(\lambda)}|, \qquad 1 \leqslant \lambda \leqslant l,$$

and hence

$$\sum_{i \notin Z} sgn(x_i - u_i^*) \cdot \sum_{\lambda=1}^{l} \alpha_\lambda u_i^{(\lambda)} \leqslant \sum_{i \in Z} \sum_{\lambda=1}^{l} \alpha_\lambda \, |u_i^{(\lambda)}| = \sum_{i \in Z} \left| \sum_{\lambda=1}^{l} \alpha_\lambda u_i^{(\lambda)} \right|.$$

The last equality holds because all summands have the same sign or vanish. By rank$[{}^U_J] = r$ and $J \subset Z$ the elementary vectors $u^{(1)}, ..., u^{(l)}$ in U are uniquely determined, and hence $u = \sum_{\lambda=1}^{l} \alpha_\lambda u^{(\lambda)}$. The classical Kolmogorov criterion is satisfied and consequently, u^* in $P_U(x)$. ▮

REFERENCES

1. P. Bloomfield and W. L. Steiger, "Least Absolute Deviations," Birkhäuser, Stuttgart, 1983.
2. M. Finzel, Linear approximation in ℓ_n^∞, *J. Approx. Theory* **76** (1994), 326–350.
3. M. Finzel, Best linear approximation in $l_1(n)$, *East J. Approx.* **2** (1996), 1–30.
4. M. Finzel, Plücker–Graßmann coordinates and elementary vectors, *in* "Approximation Theory VIII, Vol. 1, Approximation and Interpolation" (C. K. Chui and L. L. Schumaker, Eds.), pp. 207–214, 1995.
5. A. M. Pinkus, "On L^1-Approximation," Cambridge Univ. Press, Cambridge, UK, 1989.
6. C. T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, NJ, 1970.

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